# SATURATION PROPERTIES OF ULTRAFILTERS IN CANONICAL INNER MODELS

TOM BENHAMOU

Abstract. We improve Galvin's Theorem for ultrafilters which are p-point limits of p-points. This implies that in all the canonical inner models up to a superstrong cardinal, every $\kappa$-complete ultrafilter over a measurable cardinal $\kappa$ satisfies the Galvin property. On the other hand, we prove that supercompact cardinals always carry non-Galvin $\kappa$-complete ultrafilters. Finally, we prove that $\diamond(\kappa)$ implies the existence of a $\kappa$-complete filter which extends the club filter and fails to satisfy the Galvin property. This answers questions [7, Question 5.22],[3, Question 3.4] and questions ,[6, Question 4.5],[5, Question 2.26]
.

## 0. introduction

F. Galvin discovered the following remarkable saturation property of normal filters [2]: If $\kappa^{<\kappa}=\kappa$, then every normal filter $U$ over $\kappa$ satisfies the *Galvin property*, that is:

$$\forall\langle A_i \mid i<\kappa^+\rangle\in[U]^{\kappa^+}\exists I\in[\kappa^+]^{\kappa},\bigcap_{i\in I}A_i\in U.$$

A filter which satisfies the Galvin property is called a *Galvin filter*. This property, as well as some of its variants, have been recently studied in several papers [9, 10, 11, 6, 4, 5, 8, 3]. It has been used by Gitik and the author to extend the study of intermediate models of Magidor-Radin forcing [7] and the Tree-Prikry forcing [8].

Glavin's theorem was recetly improved [7] and from the same assumption, namely $\kappa^{<\kappa}=\kappa$, we can deduce that every finite product of $p$-points ultrafilters is a Galvin ultrafilter. In section 1, we improve even more this result and extend the class of filters for which Galvin's property holds to sums of $p$-points:

**Theorem 1.12.** Suppose that $W\equiv_{RK}\Sigma_U U_\alpha$ where $U$ is a $p$-point ultrafilter over $\kappa$ and for each $\alpha<\kappa$, $U_\alpha$ is a $p$-point ultrafilter over some $\delta_\alpha\leq\kappa$. Then $W$ has the Galvin property.

Although this seems as a slight improvement to the one which appears in [7], it turns out that it is significant to the analysis of ultrafilters in canonical inner models. This result easily generalizes to filters $W$ which are Rudin-Keisler equivalent to a $p$-point limit of $p$-point limits of $p$-points, and this can be pushed to any finite number of limits of $p$-points.

---

*Date*: December 28, 2022.

2010 *Mathematics Subject Classification.* 03E02, 03E35, 03E55.

*Key words and phrases.* Inner models, Galvin's property, Ultrafilters, p-point, diamond, independent family,supercompact cardinals.

0.1. **The Galvin property in canonical inner models.** In [7], it was noticed that in $L[U]$, every $\kappa$-complete ultrafilter satisfies the Galvin property. The reason is essentially that every $\kappa$-complete ultrafilter over $\kappa$ is Rudin-Keisler isomorphic to a finite product of the unique normal ultrafilter $\mathcal{U}$ which exists in $L[\mathcal{U}]$, then the theorem about finite product of $p$-points applies. Our aim here is to extend this result to canonical inner models which are compatible with greater large cardinals. More precisely, we will prove that in all the canonical inner models with no superstrong cardinal, every $\kappa$-complete ultrafilter is Rudin-Keisler isomorphic to a finite limit of $p$-points, which then falls under our improvement of Galvin's theorem. We will use G.Goldebrg's *Ultrapower Axiom*(UA) who proved [12] that this axiom should hold in every canonical inner model of set theory. Goldberg proved that UA has severe impact of the structure of $\sigma$-complete ultrafilter and this is precisely why it fits so well to this paper. We will prove the following theorem:

**Corollary 2.11.** In all canonical inner models up to a superstrong cardinal, every $\kappa$-complete ultrafilter over $\kappa$ satisfies $Gal(U, \kappa, \kappa^+)$.

At the moment, we do not know whether this result can be further pushed to canonical inner models which are compatible with greater large cardinals, and as we will see in next, relates to the open problem of finding a canonical inner model with a supercompact cardinal.

0.2. **Non-Galvin ultrafilters at very large cardinals.** In [7], it was asked if it is consistent to have a $\kappa$-complete ultrafilter which fails to satisfy the Galvin property. This was answered by Garti, Shelah and the author in [6] starting from a supercompact cardinal and improved later by Gitik and the author to a single measurable. In [6, Question 4.5], the following question appears:

**Question 0.1.** Is it consistent to have a supercomapct cardinal $\kappa$ such that every $\kappa$-complete ultrafilter satisfies the Galvin property?

This question was further investigated in [5], where it was proven that it is it consistent that every ground model $\kappa$-complete ultrafilter extends to a Galvin $\kappa$-complete ultrafilter. However, this is not enough as new $\kappa$-complete ultrafilters which do not extend ground model ultrafilters might appear in the generic extension. The third objective of this paper is to give a negative answer to this question:

**Theorem 3.1.** Suppose that $j : V \to M$ is an elementary embedding with $crit(j) = \kappa$, $M^\kappa \subseteq M$ $i_U : V \to M_U$ is the derived normal ultrapower and $k : M_U \to M$ is the factor map. Suppose that $j[\kappa^+], k[i_U(\kappa)^+] \in M$ then there is a non-Galvin $\kappa$-complete ultrafilter.

In particular, if $\kappa$ is a $2^\kappa$-supercompact cardinal then there is a non-Galvin $\kappa$-complete ultrafilter which extends the club filter. In order to achieve this we will combine three techniques, the first is the construction of a special $\kappa$-independent family from a paper by Y.Hayut [13]. Secondly we will use this independent family to construct non-Galvin filters which appears in [3] and the finally the techniques from [8] to lift elementary embedding so that a specific $\kappa$-independent family (namely, mutually generic Cohen sets) witnesses the failure of the Galvin property.

0.3. **Non-Galvin filters extending the club filter.** By applying Galvin's theorem to the club filter $Cub_\kappa$, where $\kappa$ satisfies $\kappa^{<\kappa} = \kappa$, we conclude that $Cub_\kappa$

satisfies the Galvin property, namely, that from every collection $\mathcal{A}$ of $\kappa^+$-many clubs at $\kappa$, one can always extract a subcollection $\mathcal{B} \subseteq \mathcal{A}$ of size $\kappa$ such that $\bigcap \mathcal{B}$ is a club. In order to extend Galvin's theorem, we can start by removing the assumption that $\kappa^{<\kappa} = \kappa$. In this direction, U.Abraham and S.Shelah [1] forced a model where $\kappa^{<\kappa} > \kappa$ and $Cub_\kappa$ in a non-Galvin filter, thus proving that the assumption $\kappa^{<\kappa} = \kappa$ is necessary. Another possible extension of the theorem would be to keep the assumption that $\kappa^{<\kappa} = \kappa$, and give up the normality assumption. Theorem 1.12, is exactly this kind of extension to limits of $p$-point filters. It is impossible to extend the theorem to $q$-points since a non-Galvin $\kappa$-complete ultrafilter which extends the club filter consistently exists [8]. If we do not assume large cardinals, then it does not make sense to ask for non-Galvin $\kappa$-complete ultrafilters. Instead, we can ask for non-Galvin $\kappa$-complete filters, or more interestingly ones which extend the club filter. This was specifically asked in [7, Question 5.22]. In [3], such a filter was constructed from a $\kappa$-independent family, but the filter constructed was not guaranteed to extend the club filter. The construction in this paper provides such a filter assuming that $\diamond$ holds:

**Theorem 4.6.** Suppose that $\kappa$ is a regular cardinal such that $\diamond(\kappa)$ holds, then there is a $\kappa$-complete filter extending the club filter failing to satisfy the Galvin property. Suppose that $\kappa$ is a regular cardinal and that $\diamond(\kappa)$ holds. Then there is a non-Galvin $\kappa$-complete filter $\mathcal{F}$, such that $Cub_\kappa \subseteq \mathcal{F}$.

This theorem immediately applies to $L$:

**Corollary 4.7.** $L$ is a model of $GCH$ where every regular cardinals $\kappa$ admits a non-Galvin $\kappa$-complete filter $\mathcal{F}$ which extends the club filter.

We also apply this construction of a special $\kappa$-independent family to get an ultrafilter on a supercompact cardinal which extend the club filter, this is done in theorem 4.5.

**Notations.** If $f : A \to B$ is a function we denote by $f[X] = \{f(x) \mid x \in X\}$ and $f^{-1}[Y] = \{a \in A \mid f(a) \in Y\}$. For a $\kappa$-complete ultrafilter $U$, we denote by $M_U$ the transitive collapse of the ultrapower by $U$ and $j_U : V \to M_U$ the usual ultrapower embedding. If $W, U$ are ultrafilter on $X, Y$ resp. we define the *Rudin-Keisler* $U \leq_{RK} W$ if there is a function $\pi : X \to Y$ such that for every $U = \{B \subseteq Y \mid \pi^{-1}[B] \in W\}$. If $M$ is any model of $ZFC$, the relativization of the objects to this model are denoted by $()^M$, for example $(\kappa^+)^M$, $V^M_\kappa$, $j^M_E$, etc. We say that $\kappa$ is a *Measurable cardinal* if it carries a $\kappa$-complete ultrafilter. We say it is a $\lambda$-*supercompact cardinal* if there is an elementary embedding $j : V \to M$ with $crit(j) = \kappa$ and $M^\lambda \subseteq M$. We say that it is a *Supserstrong cardinal* if there is an elementary embedding $j : V \to M$ with $crit(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$.

## 1. The Galvin property at limits of ultrafilters

Recall that an ultrafilter $U$ over $\kappa$ is called a *p-point* if for every function $f : \kappa \to \kappa$ which is not constant $(mod\ U)$, there exists $X \in U$ such that for each $\gamma < \kappa$, $f^{-1}[\gamma] \cap X$ is bounded. The following can be found in Kanamori's paper [15]:

**Proposition 1.1.** *$U$ is p-point iff for every $\alpha < j_U(\kappa)$ there is $f : \kappa \to \kappa$ such that $\alpha \leq j_U(f)(\kappa)$.*

*Proof.* Suppose that $U$ is a $p$-point. Let $\kappa \leq [f]_U < j_U(\kappa)$ also let $[\pi]_U = \kappa$. Then $\pi, f : \kappa \to \kappa$ and $f, \pi$ are not constant. Since $U$ is $p$-point we conclude that there is $X \in \mathbb{U}$ such that for every $\gamma < \kappa$, $X \cap f^{-1}[\gamma], X \cap \pi^{-1}[\gamma]$ are bounded in $\kappa$. In particular, $j_U(\pi)([id]_U) = [\pi]_U = \kappa$, Define

$$g(\gamma) = \sup(f[\sup(\pi^{-1}[\gamma+1] \cap X) + 1])$$

then $g : \kappa \to \kappa$. Let us prove that $j_U(g)(\kappa) \geq [f]_U$. Indeed,

$$j_U(g)(\kappa) = \sup(j_U(f)[j_U(\pi)^{-1}[\kappa+1] + 1) \geq j_U(f) \geq j_U(f)([id]_U) = [f]_U.$$

In the other direction, let $f$ be nonconstant. By assumption, there is a monotone function $g : \kappa \to \kappa$ such that $j_U(g)(\kappa) \geq [id]_U$. By monotonicity and since $[f]_U \geq \kappa$, $[id]_U \leq j_U(g)(\kappa) \leq j_U(g)([f]_U)$. In particular, we have that

$$X = \{\nu < \kappa \mid \nu \leq g(f(\nu))\} \in U$$

Let us prove that $f$ is almost $1-1$. Let $\gamma < \kappa$. Define $\Gamma = \sup(g[\gamma])$. The for every $\alpha \in X \cap f^{-1}[\gamma]$, $\alpha \leq g(f(\alpha)) \in g[\gamma]$. Hence $\alpha \leq \Gamma$. Hence $f^{-1}[\gamma] \cap X \subseteq \Gamma$. as wanted. $\square$

In this section, we aim to expand the class of ultrafilters which are known to be Galvin. The best result known so far is the one from [7, Corollary 5.29] where the following was proven:

**Proposition 1.2.** *Suppose that $\kappa^{<\kappa} = \kappa$ and let $F$ be a product of p-point filters over $\kappa$ . Let $\langle X_i \mid i < \kappa^+\rangle$ be a sequence of sets such that for every $i < \kappa^+$, $X_i \in F$, and let $\langle Z_i \mid i < \kappa^+\rangle$ be any sequence of subsets of $\kappa$. Then there is $Y \subseteq \kappa^+$ of cardinality $\kappa$, such that*

(1) $\bigcap_{i \in Y} X_i \in F$.
(2) *there is $\alpha \notin Y$ such that* $[Z_\alpha]^{<\omega} \subseteq \bigcup_{i\in Y}[Z_i]^{<\omega}$

*In particular, $F$ is a Galvin filter.*

**Definition 1.3.** Let $U$ be an ultrafilter over $\kappa$ and $\langle U_\alpha \mid \alpha < \kappa\rangle$ be a sequence of ultrafilters such that $U_\alpha$ is an ultrafilter over some $\delta_\alpha \leq \kappa$. Define the $U-$sum of the $U_\alpha$ as the ultrafilter $\sum_U U_\alpha$ over $\kappa \times \kappa$ defined as follows:

$$X \in \sum_U U_\alpha \Leftrightarrow \{\alpha < \kappa \mid (X)_\alpha \in U_\alpha\} \in U$$

Where $(X)_\alpha = \{\beta < \delta_\alpha \mid \langle \alpha, \beta\rangle \in X\}$.

Also, denote the $U$-limit of the $U_\alpha$'s as the ultrafilter $lim_U U_\alpha$ over $\kappa$ which is defined as follows: $X \in lim_U U_\alpha \Leftrightarrow \{\alpha < \kappa \mid X \cap \delta_\alpha \in U_\alpha\} \in U$.

*Fact* 1.4. The projection on the right coordinate $\pi_2 : \kappa \times \kappa \to \kappa$ is a Rudin-Keisler projection of $\sum_U U_\alpha$ to $lim_U U_\alpha$.

*Fact* 1.5. If $U \leq_{RK} W$, then $W$ is Galvin $\Rightarrow U$ is Galvin.

*Fact* 1.6. If $U$ is an ultrafilter over $\kappa$ and $U_\alpha$ are ultrafilters for $\alpha < \kappa$ then $j_{\sum_U U_\alpha} : V \to M_{\sum_U U_\alpha}$ can be factored to $j^{M_U}_{U^*} \circ j_U$ where $U^* = [\alpha \mapsto U_\alpha]_U$.

The following theorem is a joint result with M. Gitik:

**Theorem 1.7.** *Suppose that $W \equiv_{RK} \Sigma_U U_\alpha$ is such that $U, U_\alpha$ are p-point ultrafilters over $\kappa$. Then $W$ has the Galvin property.*

*Proof.* Let $\langle A_i \mid i<\kappa^+\rangle \in [W]^{\kappa^+}$, denote by $A^{(1)}_{i,\alpha}=\{\beta \mid \langle\alpha,\beta\rangle\in A_i\}$. Also, let $A^{(0)}_i=\{\alpha \mid A^{(1)}_{i\alpha}\in U_\alpha\}\in U$. For every $\alpha_1<\alpha_2\in[\kappa]^2$, and every $\xi<\kappa^+$, define

$$H_{\xi,\langle\alpha_1,\alpha_2\rangle}=\Big\{\gamma<\kappa^+ \mid .A^{(0)}_\gamma\cap\alpha_1=A^{(0)}_\xi\cap\alpha_1 \text{ and } \forall\beta<\alpha_1.A^{(1)}_{\gamma,\beta}\cap\alpha_2=A^{(1)}_{\xi,\beta}\cap\alpha_2\Big\}.$$

**Claim 1.8.** *There is $\xi^*<\kappa^+$, such that for every $\alpha_1,\alpha_2$, $|H_{\xi,\langle\alpha_1,\alpha_2\rangle}|=\kappa^+$.*

*Proof of Claim.* Suppose otherwise, then pick for each $\xi<\kappa^+$, $\alpha_{1,\xi},\alpha_{2,\xi}<\kappa$ such that the cardinality is at most $\kappa$. Stabilize these values on a set $X$ of size $\kappa^+$ with the values $\alpha_1^*,\alpha_2^*$. The set $H_{\xi,\langle\alpha_1^*,\alpha_2^*\rangle}$ is determined from $A^{(0)}_\xi\cap\alpha_1^*$ and $\langle A^{(1)}_{\xi,\beta}\cap\alpha_2^* \mid \beta<\alpha_1^*\rangle$. Since there are less than $\kappa$ many such sequences, we have a set $X^*$ of size $\kappa^+$ such that for any $\xi\in X^*$, $H_{\xi,\langle\alpha_1^*,\alpha_2^*\rangle}=H$ but then $X^*\subseteq H$, contradiction. $\square$

*End of proof of Theorem 1.7.* Since all the ultrafilters are $p$-points, for each $W\in\{U\}\cup\{U_\alpha \mid \alpha<\kappa\}$ there is a set $B^{(W)}\in W$ such that for every $j<\kappa$, there is

$$\rho^{(j)}_W>\sup(\pi_W^{-1}[j]\cap B^{(W)}),j$$

where $[\pi_W]_W=\kappa$. For $\alpha,j<\kappa$, let $\delta^{(j)}_\alpha=\sup(\rho^{(j)}_{U_\beta} \mid \beta<\alpha)<\kappa$. Define the sequence $\beta_j$ by induction,

$$\beta_j\in H_{\alpha^*,\langle\rho^{(j)}_U,\delta^{(j)}_{\rho^{(j)}_U}\rangle}\setminus\{\beta_k \mid k<j\}.$$

We claim that

$$\textstyle\bigcap_{j<\kappa}A_{\beta_j}\in\Sigma_U U_\alpha$$

To see this, set

$$C^{(0)}:=A^{(0)}_{\alpha^*}\cap\Delta^*_{j<\kappa}A^{(0)}_{\beta_j}\cap B^{(U)}\in U.$$

and for each $\beta\in C^{(0)}$ let

$$C_\beta=\Big(A^{(1)}_{\alpha^*,\beta}\cap\Delta^*_{j<\kappa}A^{(1)}_{\beta_j,\beta}\cap B^{(U_\beta)}\Big)\setminus\rho^{(\beta)}_{U_\beta}\in U_\beta.$$

Finally we let $C^*=\bigcup_{\beta\in C^{(0)}}\{\beta\}\times C_\beta$. Clearly, $C^*\in U-\Sigma_\beta U_\beta$. Let $\langle\alpha_1,\alpha_2\rangle\in C^*$, then $\alpha_1\in C^{(0)}$ and $\alpha_2\in C_{\alpha_1}$. Let $j<\kappa$, if $j<\pi_U(\alpha_1)$ then $\alpha_1\in A^{(0)}_{\beta_j}$. If $\pi_U(\alpha_1)\le j$, then $\alpha_1<\rho^{(j)}_U$, so $\alpha_1\in A^{(0)}_{\alpha^*}\cap\rho^{(j)}_U$. Since $\beta_j\in H_{\alpha^*,\langle\rho^{(j)}_U,\dots,\rho^{(j)}_n\rangle,\vec{\nu}_j}$, $\alpha_i\in A^{(0)}_{\beta_j}\cap\rho^{(j)}_U$. We conclude that in any case $\alpha_1\in A^{(0)}_{\beta_j}$. By definition of $C_{\alpha_1}$, $\alpha_2>\rho^{(\alpha_1)}_{U_{\alpha_1}}$, which implies that $\pi_{U_{\alpha_1}}(\alpha_2)>\alpha_1$. If $j<\pi_{U_{\alpha_1}}(\alpha_2)$, then $\alpha_2\in A^{(1)}_{\beta_j,\alpha_1}$, by definition of the modified diagonal intersection, and moreover, $\langle\alpha_1,\alpha_2\rangle\in A_{\beta_j}$. If $\pi_{U_{\alpha_1}}(\alpha_2)\le j$, then $\alpha_1<j<\rho^j_U$ and thus $\alpha_2<\rho^{(j)}_{U_{\alpha_1}}<\delta^{(j)}_{\rho^{(j)}_U}$. since $\beta_j\in H_{\alpha^*,\rho^{(j)}_U,\delta^{(j)}_{\rho^{(j)}_U}}$, we conclude that $\alpha_2\in A^{(1)}_{\alpha^*,\alpha_1}\cap\rho^{(j)}_{U_{\alpha_1}}=A^{(1)}_{\beta_j,\alpha_1}\cap\rho^{(j)}_{U_{\alpha_1}}$ hence $\langle\alpha_1,\alpha_2\rangle\in A_{\beta_j}$. This establishes the proof that $C^*\subseteq\bigcap_{j<\kappa}A_{\beta_j}$. $\square$

The previous argument generalizes to ultrafilters which a obtained by taking limits of p-point finitely many times.

**Corollary 1.9.** *Suppose $W\equiv_{RK} lim_U U_\alpha$ where $U,U_\alpha$ are all $p-$points on $\kappa$, then $W$ is Galvin.*

*Proof.* The projection to the right coordinate $\pi_2 : \kappa \times \kappa \to \kappa$ is a Rudin-Keisler projection from $\sum_U U_\alpha$ to $lim_U U_\alpha$. The Galvin property is downward preserved in the Rudin-Keisler order. □

Next, we deal with the case that the critical point of the ultrafilters $U_\alpha$ is not necessarily $\kappa$. Denote by $\delta_\alpha \leq \kappa$ the ordinal for which $U_\alpha$ is an ultrafilter over and $\pi_U : \kappa \to \kappa$ the function representing $\kappa$ in $M_U$. First note that on a there is a set $X \in U$ such that exactly one of the following holds:

(1) For each $\alpha \in X$, $\delta_\alpha = \kappa$.
(2) For each $\alpha \in X$, $\pi_U(\alpha) < \delta_\alpha < \kappa$
(3) For each $\alpha \in X$, $\pi_U(\alpha) = \delta_\alpha$

*Fact* 1.10. If $U, U_\alpha, U'_\alpha$ are ultrafilters such that $\{\alpha \mid U_\alpha \neq U'_\alpha\} \notin U$, then $\sum_U U_\alpha = \sum_U U'_\alpha$.

**Lemma 1.11.** *Suppose that $U, U_\alpha$ p-point ultrafilters such that for every $\alpha < \kappa$, $U_\alpha$ is an ultrafilter over $\pi_U(\alpha) < \delta_\alpha < \kappa$. Then $\sum_U U_\alpha \equiv_{RK} W$ where $W$ is a p-point ultrafilter.*

*Proof.* Let $W$ be any ultrafilter over $\kappa$ such that $\sum_U U_\alpha =_{RK} W$, let us prove that $W$ is a $p$-point. Note that by Rudin-Keisler equivalence, $M_W = M_{\sum_U U_\alpha}$ and $j_W = j_{\sum_U U_\alpha}$. Let $U^* = [\alpha \mapsto U_\alpha]_U$, then $U^* \in M_U$ and $U^*$ is an ultrafilter over

$$\kappa = [\pi_U]_U < \kappa^* = [\alpha \mapsto \delta_\alpha]_U < j_U(\kappa).$$

By fact 1.5, $j_W = j_{\sum_U U_\alpha}$ can be factored into $j_{U^*} \circ j_U$. Now since $j_U(\kappa)$ is a measurable cardinal in $M_U$, and $\kappa^* = crit(j_{U^*}) < j_U(\kappa)$, it follows that $j_W(\kappa) = j_{U^*}(j_U(\kappa)) = j_U(\kappa)$. To see that $W$ is a $p$-point, let us use Lemma 1.1, let $f : \kappa \to \kappa$ be any function such that $[f]_W < j_W(\kappa) = j_U(\kappa)$. Since $U$ is a $p$-point, find some monotone function $g : \kappa \to \kappa$ such that $[f]_W \leq j_U(g)(\kappa)$. It follows that $[f]_W \leq j_{U^*}(j_U(g)(\kappa)) = j_W(g)(j_{U^*}(\kappa))$. Finally, since $\kappa^* > \kappa$ it follows that $j_{U^*}(\kappa) = \kappa$, so $[f]_W \leq j_W(g)(\kappa)$ as wanted. □

**Theorem 1.12.** *Suppose that $W \equiv_{RK} \Sigma_U U_\alpha$ where $U$ is a p-point ultrafilter over $\kappa$ and for each $\alpha < \kappa$, $U_\alpha$ is a p-point ultrafilter over some $\delta_\alpha \leq \kappa$. Then $W$ has the Galvin property.*

*Proof.* By the observation above, we may assume that either for every $\alpha < \kappa$ $\delta_\alpha = \kappa$, in which case we can apply Theorem 1.9. The second case, where $\pi_U(\alpha) < \delta_\alpha < \kappa$, follows from the previous lemma. Finally, if $\pi_U(\alpha) = \delta_\alpha$, then $U^* = [\alpha \mapsto U_\alpha]_U$ is a $p$-point ultrafilter over $[\pi_U]_U = \kappa$ in $M_U$. Since $M_U$ is closed under $\kappa$-sequences, it follows that $U^*$ is $p$-point over $\kappa$ is $V$. So $\sum_U U_\alpha \equiv_{RK} U^* \times U$, and by theorem 1.2, a product of $p$-points is Galvin. □

## 2. Galvin ultrafilters in canonical inner models

In this section, we wish to apply the results from the previous section to study the Galvin property in canonical inner models. The first result in this direction was done in [7]:

**Proposition 2.1.** *Let $U$ be a normal measure over $\kappa$. Then in $L[U]$, every $\kappa$-complete (even $\sigma$-complete) ultrafilter is a Galvin ultrafilter.*

The reason for this is that every $\kappa$-complete ultrafilter in $L[U]$ is Rudin-Keisler equivalent to a finite power of the normal measure $U$, and then theorem 1.2 applies to such ultrafilters. Our intention is to use a similar argument to prove the same result for canonical inner models suitable for larger cardinals. It turns out that the results from the previous section can be used to prove that in every canonical inner model up to a superstrong cardinal, every $\kappa$-complete ultrafilter is Galvin.

In order to prove that, we will use Goldberg's recent work about the *Ultrapower Axiom* UA [12] and use his results regarding the structure of ultrafilters under this axiom.

Let us start this section by stating Goldberg's results which are going to be used here. First, and most importantly, the fact the UA follows from Weak Comparison, and therefore should hold in all canonical inner models:

**Theorem 2.2** (Theorem 2.3.10. [12]). *Assume that $V = HOD$ and there is a $\Sigma_2$-correct worldly cardinal. If Weak Comparison holds, then the Ultrapower Axiom holds*

The UA has many consequences which relate to the structure of $\kappa$-complete ($\sigma$-complete) ultrafilters. The one which is relevant to our work is the factorization to irreducible ultrafilters.

**Definition 2.3.** Let $U, W$ be $\sigma$-complete ultrafilters. We define the Rudin-Frulík ordering of ultrafilter by $U \leq_{RF} W$ is there is there is a set $I \in U$ and a discrete sequence of ultrafilters $\langle W_i : i \in I\rangle$ such that $W = lim_U Wi$ .

There is an equivalent formulation (for $\sigma$-complete ultrafilters) in terms of ultrapowers, $U \leq_{RF} W$ if and only if there is an internal ultrapower embedding $i : M_U \to M_W$ such that $i \circ j_U = j_W$.

**Definition 2.4.** A $\sigma$-complete ultrafilter $W$ is called *irreducible*, if whenever $U \leq_{RF} W$ then either $U \simeq W$ or $U$ is principle. Equivalently, $W$ is minimal in the Rudin-Frulík order among the non-principle ultrafilters.

The notion of irreducible ultrafilters was used by Goldberg to describe the $\sigma$-complete ultrafilters under UA:

**Theorem 2.5** (Theorem 5.3.16[12]). *Assume* UA. *Then for every $\sigma$-complete ultrafilter $W$, there is a finite linear iterated ultrapower $\langle M_n, U_m, j_{m,n} : m < n \leq l\rangle$ such that $M_0 = V, M_l = M_W$ , and $U_m$ is an irreducible ultrafilter of $M_m$ for all $m < l$, and $j_W = j_{0,l}$.*

**Corollary 2.6.** *Assume* UA*, then every $\sigma$-complete ultrafilter $W$ has the form*

$$W = \sum_U (\sum_{\sum_{\alpha_1}} ...( \sum_{U_{\alpha_1,\dots,\alpha_{n-1}}} U_{\alpha_1,\dots,\alpha_n})))$$

*where each $U_{\alpha_1,..,\alpha_k}$ is irreducible.*

Next we will also need Schlutzenberg [16] result about $\sigma$-complete ultrafilters in the models $L[\mathbb{E}]$:

**Theorem 2.7** (Theorem 4.3.2 [12]). *Suppose $L[\mathbb{E}]$ is an iterable Mitchell-Steel model and $U$ is a $\sigma$-complete ultrafilter of $L[\mathbb{E}]$. Then the following are equivalent:*

1. *$U$ is irreducible.*
2. *$U$ is isomorphic to a Dodd sound ultrafilter.*

(3) *U is isomorphic to an extender on the sequence* $\mathbb{E}$.

**Theorem 2.8.** *Suppose that there is no inner model with a superstrong cardinal, then in* $L[\mathbb{E}]$ *every irreducible ultrafilter is p-point.*

*Proof.* Suppose not, and let $f$ be the $U$-minimal almost one-to-one function. Let $\nu = [f]_U$:

**Claim 2.9.** $\kappa < \nu$ *is a generator, namely, for every* $g : \kappa \to \kappa$ *and every* $\rho < \nu$, $j_U(g)(\rho) < \nu$.

*Proof of claim.* Indeed, if for some $\rho < \nu$ and some increasing $g$, $j_U(g)(\rho) \geq [f]_U$, let $\rho = [h]_U$, then $[gh]_U \geq [f]_U$ let us prove that $[h]_U$ is almost one-to-one which will contradict the minimality of $[f]_U$. We have a set $X \in U$ such that for each $\alpha \in X$:

(1) $g(h(\alpha)) \geq f(\alpha)$.
(2) $h(\alpha) < f(\alpha)$.

Let $\lambda < \kappa$, if $h(\alpha) < \lambda$, then $g(h(\alpha)) < g(\lambda)$ and therefore $f(\alpha) \leq g(\lambda)$. Hence $h^{-1}[\lambda] \subseteq f^{-1}[g(\lambda)]$. Since $f$ is almost one-to-one we have that $f^{-1}[g(\lambda)]$ is bounded and also $h^{-1}[\lambda]$ is bounded. □

Let $E$ be the $(\kappa, \nu)$-extender derived from $\nu$ and $j_U$, then $j_E(\kappa) = \nu$. Indeed, $j_E(\kappa) \geq \nu$ and $crit(k) \geq \nu$, where $k : M_E \to M_U$ is the factor map (as for any extender of length $\nu$). Suppose toward a contradiction that $j_E(\kappa) > \nu$, then pick any $\nu < \gamma < j_E(\kappa)$ and there is $a \in [\nu]^{<\omega}$ and $g : [\kappa]^{|a|} \to \kappa$ such that $j_E(g)(a) = \nu$ and we can define $g' : \kappa \to \kappa$ by $g'(\alpha) = \sup_{b \in [\alpha]^{|a|}} g(b)$ to see that $j_E(g')(\rho) \geq \nu$ for $\rho = \max(a) + 1 < \nu$. Now apply $k$ to get $j_U(g')(\rho) \geq k(\nu) \geq \nu$, contradiction the fact that $\nu$ is a generator of $j_U$. Hence $j_E(\kappa) = \nu$.

By Theorem 2.7, $U$ is isomorphic to some extender $E^*$ in $\mathbb{E}$ and $j_U = j_{E^*}$. The extender $E$ is just $E^* \restriction \nu$ and therefore by the initial segment condition, $E \in M_U$ (see [17]) now form $Ult(M_U, E)$, then we claim that $(V_\nu)^{M_U} \subseteq (M_E)^{M_U}$, namely, every element $x \in M_U$ with $rank(x) < \nu$ is a member of $(M_E)^{M_U}$. Indeed, consider $j_E \restriction M_U : M_U \to j_E(M_U)$. Since $P(\kappa) \subseteq M_U$, we have that $E$ is the $(\kappa, \nu)$-extender derived from $j_E \restriction M_U$, hence for every $a \in [\nu]^{<\omega}$, there is a factor map $k_a : (M_{E_a})^{M_U} \to j_E(M_U)$ defined by $k_a([f]_{E_a}) = j_E(f)(a)$ such that $k_a \circ (j_{E_a})^{M_U} = j_E \restriction M_U$. Since the factor maps $k_a$ commutes with the projections, it follows that there is a factor map $k : (M_E)^{M_U} \to j_E(M_U)$ and $j_E \restriction M_U = k \circ (j_E)^{M_U}$. Now for every $\rho < \nu$, consider $[id]_{E_\rho} \in (M_{E_a})^{M_U}$ and thus $j_{\rho,\infty}([id]_{E_\rho}) = x \in (M_E)^{M_U}$ and we have that

$$k(x) = k(j_{\rho,\infty}([id]_{E_\rho})) = k_\rho([id]_{E_\rho}) = j_E(id_{E_\rho})(\rho) = \rho$$

Hence $\nu \subseteq Im(k)$ and therefore $crit(k) \geq \nu$ and in particular for every $x \in M_U$ of rank $< \nu$, $k(x) = x \in (M_E)^{M_U}$.

We conclude that in $M_U$ we have a $(\kappa, \nu)$-extender $E$ such that $j_E(\kappa) = \nu$ and $(M_E)^{M_U}$ includes all the elements of $M_U$ of rank $< j_E(\kappa)$. So $M_U \models \kappa$ is a superstrong cardinal which is a contradiction. □

**Corollary 2.10.** *Suppose that there is no inner model with a strong cardinal* $U$ *is a* $\kappa$*-complete ultrafilter over* $\kappa$ *in* $L[\mathbb{E}]$ *then* $U$ *is Rudin-Keisler isomorphic to a filter of the form*

$$\sum_U \sum_{U_{\alpha_1}} \cdots \sum_{U_{\alpha_1,\ldots,\alpha_{n-1}}} (U_{\alpha_1,\ldots,\alpha_n})$$

*where each* $U_{\alpha_1,...,\alpha_k}$ *for* $0 \leq k \leq n$ *is a p-point.*

**Corollary 2.11.** *In all canonical inner models up to a superstrong cardinal, every* $\kappa$*-complete ultrafilter satisfies* $Gal(U, \kappa, \kappa^+)$.

**Question 2.12.** Does UA imply that every $\kappa$-complete ultrafilter over $\kappa$ satisfies that Galvin property?

We conjecture that the answer is no. However, is it still interesting to ask how far in the hierarchy of inner models we have that every $\kappa$-complete ultrafilter over $\kappa$ satisfies the Galvin property?

**Question 2.13.** Is there a Galvin ultrafilter which is not Rudin-Keisler equivalent to an ultrafilter of the form appearing in corollary 2.10?

In terms of ultrapowers, an ultrafilter $W$ which is not of Rudin-Keisler equivalent to an ultrafilter of the form appearing in corollary 2.10, should have infinitely many generators with respect to $j_W$. Alternatively, one can use Kanamori's terminology and require that $W$ has infinitely many skies.

## 3. Supercompact cardinals

So far we have established that in the canonical inner models up to a superstrong, every $\kappa$-complete ultrafilter has the Galvin property. This shows that even for larger cardinals than just a measurable, there is no $ZFC$ construction of a $\kappa$-complete ultrafilter which is not Galvin. However, if we turn to the realm of large cardinals where we currently have no canonical inner models, this situation changes. Below we prove that there is a $ZFC$ construction for a non-Galvin ultrafilter over $\kappa$ assuming that $\kappa$ is a supercompact cardinal. This answer a question from [6, Question 4.5] and the parallel one in [5, Question 2.25].

**Theorem 3.1.** *Suppose that* $j : V \to M$ *is an elementary embedding with* $crit(j) = \kappa$*,* $M^\kappa \subseteq M$ $i_U : V \to M_U$ *is the derived normal ultrapower and* $k : M_U \to M$ *is the factor map. Suppose that* $j[\kappa^+], k[i_U(\kappa)^+] \in M$ *then there is a non-Galvin* $\kappa$*-complete ultrafilter.*

*Proof.* Let $\langle A_i \mid i < \kappa^+\rangle$ be a $\kappa$-independent family. Denote by $\langle A'_\alpha \mid \alpha < j(\kappa)^+\rangle = j(\langle A_i \mid i < \kappa^+\rangle)$. Since $j[\kappa^+], k[i(\kappa)^+] \in M$, in $M$ we have the sequence $\langle A'_r \mid r \in j[\kappa^+]\rangle$ and $\langle A'_s \mid s \in k[i_U(\kappa)^+] \setminus j[\kappa^+]\rangle$. Since $M \models |j[\kappa^+]| = \kappa^+ < j(\kappa)$ and $M \models |k[i_U(\kappa)^+]| = \kappa^+ < j(\kappa)$, the $\kappa$-independence of the family implies that there is

$$\kappa \leq \delta \in (\bigcap_{r\in j[\kappa^+]} A'_r) \cap (\bigcap_{s\in k[i(\kappa)^+]\setminus j[\kappa^+]} (A'_s)^c).$$

Let $W = \{X \subseteq \kappa \mid \delta \in j(X)\}$. Then $\{A_i \mid i < \kappa^+\} \subseteq W$ and for every $I \in [\kappa^+]^\kappa$, consider $i_U(I)$, there is $r \in i_U(I) \setminus i_U[\kappa^+]$ e.g the $\kappa$-th elements in $i_U(I)$ in the increasing enumeration. then $k(r) \in j(I)$ and not in $j[\kappa^+$, thus $\delta \notin A'_{k(r)}$. It follows that $\delta \notin \bigcap_{s\in j(I)} A'_s = j(\bigcap_{i\in I} A_i)$. This implies that $\bigcap_{i\in I} A_i \notin W$. □

**Corollary 3.2.** *If* $\kappa$ *is* $2^\kappa$*-supercompact then there is always a non-Galvin ultrafilter.*

**Proposition 3.3.** *The first cardinal such that there is a non-Galvin ultrafilter is below the first supercompact.*

*Proof.* Let $\kappa$ be $2^\kappa$-supercmpact, then by the previous corollary $\kappa$ carries a $\kappa$-complete ultrafilter $U$ which is not Galvin. Let $j : V \to M$ be an embedding witnessing $2^\kappa$-supercompactness, then $U \in M$ and there is a witness $\langle A_i \mid i < 2^\kappa\rangle$ for $\neg Gal(U, \kappa, \kappa^+)$ which also belongs to $M$. Therefore,

$$M \models \text{"}\kappa \text{ carries a non-Galvin } \kappa\text{-complete ultrafilter"}$$

and by reflection, this should hold at many cardinals below $\kappa$. □

## 4. Non-Galvin ultrafilters Extending the club filter

Note that the ultrafilter we constructed in the previous section does not necessarily extend the club filter. However, this was not the situation in the latest constructions of non-Galvin ultrafilters [6, 8, 3]. In order to achieve this, let us choose our independent family a bit differently. The idea is to use Y. Hayut's construction of independent families with special properties from [13]. Hayut considered these families for the purpose of proving the equivalence between the normal filter extension property and the filter extension property. Recall that a sequence of subsets of $\kappa$ $\langle A_i \mid i < \lambda\rangle$ is called $\kappa$-independent if for every $I, J \in [\lambda]^{<\kappa}$, if $I \cap J = \emptyset$ then $(\bigcap_{i\in I} A_i) \cap (\bigcap_{j\in I} A^c_j) \neq \emptyset$.

**Definition 4.1.** A sequence $\langle A_i \mid i < \lambda\rangle$ is called a *normal* $\kappa$-independent family, if it is $\kappa$-independent and for every $\langle A_{\alpha_i} \mid i < \kappa\rangle, \langle A_{\beta_i} \mid i < \kappa\rangle \subseteq \langle A_i \mid i < \lambda\rangle$ are any two disjoint collections, then $\Delta_{i<\kappa} A_{\alpha_i} \setminus A_{\beta_i}$ is a stationary subset of $\kappa$.

The following proposition is due to Hayut[1]:

**Proposition 4.2.** *If $\diamond(\kappa)$ holds then there is a normal $\kappa$-independent family of length $2^\kappa$.*

*Proof.* Let $\langle X_i \mid i < \kappa\rangle$ be $\diamond(\kappa)$-sequence. Let us start by multiplying it, by fixing a bijection $\phi : \kappa \times \kappa \times \{0, 1\} \leftrightarrow \kappa$. We fix a club $C^*$ of all $\alpha < \kappa$ such that $\phi \restriction \alpha \times \alpha \times \{0, 1\} : \alpha \times \alpha \times \{0, 1\} \leftrightarrow \alpha$. Consider $\phi^{-1}[X_\alpha]$, for $\alpha \in C^*$, we can identify this with a sequence: $\langle\langle Y^\alpha_i, Z^\alpha_i\rangle \mid i < \alpha\rangle$.

Take any sequence $\mathcal{A} = \langle\langle A_i, B_i\rangle \mid i < \kappa\rangle$. Then $\mathcal{A} \subseteq \kappa \times \kappa \times \{0, 1\}$, and we translate $\phi[\mathcal{A}] = B$. Then the set $\{\alpha \in C^* \mid B \cap \alpha = X_\alpha\}$ is stationary and for each such $\alpha$,

$$B \cap \alpha = \phi[\mathcal{A} \cap (\alpha \times \alpha \times \{0, 1\})] = \phi[\langle\langle A_i \cap \alpha, B_i \cap \alpha \mid i < \alpha\rangle]$$

Hence for each $i < \alpha$ $Y^{(\alpha)}_i = A_i \cap \alpha$ and $Z^{(\alpha)}_i = B_i \cap \alpha$.

Now suppose that $Y \subseteq \kappa$, define

$$R_Y := \{\alpha \in C^* \mid \forall i, j < \alpha. Y^{(\alpha)}_i \neq Z^{(\alpha)}_j \text{ and } \exists i < \alpha. Y \cap \alpha = Y^{(\alpha)}_i\}.$$

**Claim 4.3.** *$\langle R_Y \mid Y \subseteq \kappa\rangle$ is a normal $\kappa$-independent family of size $2^\kappa$.*

*Proof.* First, if $Y \neq Y'$ let $\nu \in Y \Delta Y'$. Consider the sequence

$$\mathcal{Y} = \langle\langle Y, Y'\rangle \mid i < \kappa\rangle.$$

Find $\alpha > \nu$ which guesses $\langle\langle Y \cap \alpha, Y' \cap \alpha\rangle \mid i < \alpha\rangle$. In particular, for each $i, j < \alpha$, $Y^{(\alpha)}_i = Y \cap \alpha \neq Y' \cap \alpha = Z^{(\alpha)}_j$ and therefore $\alpha \in R_Y \Delta R_{Y'}$.

Let $\langle R_{Y_i} \mid i < \lambda\rangle, \langle R_{Z_j} \mid j < \lambda'\rangle$ where $\lambda, \lambda' < \kappa$. For each $i < \lambda, j < \lambda'$ find $\alpha_{i,j} < \kappa$ such that $Y_i \cap \alpha_{i,j} \neq Z_j \cap \alpha_{i,j}$ and let $\alpha^* = \sup_{i,j} \alpha_{i,j} < \kappa$. Consider the

[1] Hayut uses the more general framework of stationary sets over $P_\kappa(\lambda)$ due to Jech [14].

sequence $\langle Y_i, Z_i \mid i < \kappa\rangle$ such that if $i$ is not define take $Y_i = Y_0$ or $Z_i = Z_0$. Find $\alpha$ which guesses the sequence $\langle\langle Y_i \cap \alpha, Z_i \cap \alpha\rangle \mid i < \alpha\rangle$ such that $\alpha > \lambda, \lambda', \alpha^*$. Then such $\alpha$ belongs to $\bigcap_{i<\lambda} R_{Y_i} \cap \bigcap_{j<\lambda'} (R_{Z_j})^c$.

Finally, for normality, suppose that we have $\langle Y_i, Z_i \mid i < \kappa\rangle$ as in the definition. There are stationary many $\alpha$ which guesses this sequence and clearly, they would belong to every $Y_i \setminus Z_i$ for any $i < \alpha$. □

□

**Corollary 4.4.** *In a normal $\kappa$-independent family, the intersection of any less than $\kappa$ many of the sets and less than $\kappa$ many of the complement is stationary.*

*Proof.* In the previous construction, this was clear since by the $\diamond$ we can find stationarily many $\alpha$ in each of the intersections. However, this is true also from the abstract definition, let $I, J \in [2^\kappa]^{<\kappa}$ by such that $I, J$ are disjoint, we want to prove that $(\bigcap_{i\in I} A_i) \cap (\bigcap_{j\in J} A_j^c)$ is stationary. Indeed, enumerate $I = \langle i_\alpha \mid \alpha < \lambda\rangle$ and $J = \langle j_\beta \mid \beta < \lambda'\rangle$ and define $i_\alpha = i_0$, $j_\beta = j_0$ for $\alpha \geq \lambda$ and $\beta \geq \lambda'$. By the definition of a normal $\kappa$-independent family,

$$S := \Delta_{\alpha<\kappa} A_{i_\alpha} \setminus A_{j_\alpha}$$

is stationary in $\kappa$. Let us take any $\kappa > \rho \geq \lambda, \lambda'$. It is not hard to check that

$$S \setminus \rho \subseteq (\bigcap_{i\in I} A_i) \cap (\bigcap_{j\in J} A_j^c)$$

and therefore this intersection is stationary. □

**Corollary 4.5.** *If $\kappa$ is $2^\kappa$-supercompact then there is a $\kappa$-complete ultrafilter which extends the club filter and $\neg Gal(U, \kappa, \kappa^+)$.*

*Proof.* First note that since $\kappa$ is $2^\kappa$-supercompact, then $\diamond(\kappa)$ holds. Apply Proposition 4.2 to obtain a normal $\kappa$-independent family $\langle A_i \mid i < 2^\kappa\rangle$. Next, we use the same construction as in theorem 3.1 starting with the family $\langle A_i \mid i < 2^\kappa\rangle$. Note by $2^\kappa$-closure of $M$, $j[Cub_\kappa]$ is a set of less than $j(\kappa)$ many clubs at $j(\kappa)$ and hence $\cap j[Cub_\kappa]$ is a club at $j(\kappa)$. Applying the normality of the family we can find now

$$\kappa \leq \delta \in (\bigcap j[Cub_\kappa]) \cap (\bigcap_{\alpha<\kappa^+} j(A_\alpha)) \cap (\bigcap_{\beta\in k[i(\kappa^+)]\setminus j[\kappa^+]} (A'_\beta)^c)$$

By the previous corollary, $(\bigcap_{\alpha<\kappa^+} j(A_\alpha)) \cap (\bigcap_{\beta\in k[i(\kappa^+)]\setminus j[\kappa^+]} (A'_\beta)^c)$ is stationary in $j(\kappa)$ and therefore such $\delta$ exists. Deriving the same ultrafilter $W = \{X \subseteq \kappa \mid \delta \in j(X)\}$, it is clear that $W$ is a non-Galvin $\kappa$-complete ultrafilter over $\kappa$ which extends the club filter. □

This type of independent family can further be used to answer another question [3, Question 3.4] about constructing a $\kappa^+$-complete filter $\mathcal{F}$ which extends the club filter on $\kappa^+$ and $\neg Gal(\mathcal{F}, \kappa^+, \kappa^{++})$. Here we will prove that there is such a filter if we just assume $\diamond(\kappa)$ and in particular, $L$ is a model where there is always such a filter. This demonstrates, without large cardinals, why Galvin's theorem (and in fact the generalization of it which was considered here) is sharp in the sense that we cannot drop the combinatorial assumption about the filter from Galvin's theorem.

**Theorem 4.6.** *Suppose that $\kappa$ is a regular cardinal such that $\diamond(\kappa)$ holds, then there is a $\kappa$-complete filter extending the club filter which fails to satisfy the Galvin property.*

*Proof.* Suppose that $\langle A_i \mid i < \kappa^+\rangle$ is a normal-independent family of $\kappa^+$-many subsets of $\kappa$. Let us define $\mathcal{F}$ to be the minimal $\kappa$-complete filter which extends the family $Cub_\kappa \cup \{A_i \mid i < \kappa^+\}$. Namely, we define

$$\mathcal{F} = \{X \subseteq \kappa \mid \exists C \in Cub_\kappa.\exists I \in [\kappa^+]^{<\kappa}.C \cap (\bigcap_{i\in I} A_i) \subseteq X\}$$

This is indeed a $\kappa$-complete filter since the intersection of fewer than $\kappa$ many of the sets $A_i$ is stationary; therefore, this family has the $<\kappa$-intersection property holds. This guarantees that it generates a $\kappa$-complete filter which we denote by $\mathcal{F}$. Clearly, $\mathcal{F}$ extends the club filter. Let us prove that the family $\langle A_i \mid i < \kappa^+\rangle$ witnesses the failure of the Galvin property. Suppose otherwise, then there is $I \in [\kappa^+]^\kappa$ such that $\bigcap_{i\in I} A_i \in \mathcal{F}$. It follows by the definition of $\mathcal{F}$, that there is $J \in [\kappa^+]^{<\kappa}$ and a club $C$ such that $C \cap (\bigcap_{j\in J} A_j) \subseteq \bigcap_{i\in I} A_i$. pick any $i^* \in I \setminus J$. Such an index exists since $|I| = \kappa$ and $|J| < \kappa$. By normal independence, $(\bigcap_{j\in J} A_j) \cap A^c_{i^*}$ is stationary, and therefore there is some $\nu \in C \cap (\bigcap_{j\in J} A_j)$ such that $\nu \notin A_{i^*}$. But this is absurd since this would mean that $\nu \notin \bigcap_{i\in I} A_i$. □

It follows now, unlike the situation with non-Galvin ultrafilters, that the existence of non-Galvin filters fits well in canonical inner models.

**Corollary 4.7.** *Assume $V = L$, then every regular cardinal admits a $\kappa$ complete filter extending the club filter and fails to satisfy the Galvin property.*

*Remark* 4.8. Note that the filter we constructed in Theorem 4.6, extends to a normal $\kappa$-complete filter $\mathcal{F}^*$, namely,

$$\mathcal{F}^* := \{X \subseteq \kappa \mid \exists\langle \alpha_i \mid i < \kappa\rangle \in [\kappa^+]^\kappa.\Delta_{i<\kappa} A_{\alpha_i} \subseteq X\}$$

which is again Galvin. This is quite an interesting situation as the known methods which "correct" a non-Galvin ultrafilter and makes it Galvin (see [5]) are quite brutal in the sense that we either collapse the generator of an ultrafilter so it extends to an ultrafilter which is isomorphic to a normal one or we diagonalize the filter and make it eventually a $p$-point.

A closely related question to the existence of a $\kappa$-complete filter $\mathcal{F}$ extending the club filter such that $Gal(\mathcal{F}, \kappa, \kappa^+)$ fails is the question about the $\kappa^+$-saturation of the club filter over $\kappa$ (or the dual non-stationary ideal) which is the assertion that that from every $\kappa^+$-many *stationary* sets there are always two for which the intersection is again stationary. A classical result of Shelah establishes that for any $\kappa > \omega_1$, the non-stationary ideal on $\kappa^+$ is not saturated and this result was later generalized to include inaccessible cardinals by Gitik and Shelah. In light of these results, it would not be too bold to conjecture that it is ZFC-provable that every regular cardinal admits a non-Galvin, $\kappa$-complete filter which extends the club filter.

Let us conclude this paper with one more piece of evidence, which is given below in our last proposition. The proposition also demonstrates that the diamond principle is not necessary for the existence of such a filter and even GCH is not needed.

**Proposition 4.9.** *Let $\langle A_i \mid i < \lambda\rangle$ be a sequence of $\lambda$-many mutually generic Cohen subsets of $\kappa$ over $V$, then for each $I, J \in [\lambda]^{<\kappa}$, such that $I \cap J = \emptyset$, $(\bigcap_{i\in I} A_i) \cap (\bigcap_{j\in J} A_j)$ is stationary. In particular, if $\mathcal{F}$ is the $\kappa$-complete filter* *generated by* $Cub_\kappa \cup \{A_i \mid i < \lambda\}$*, then the sequence* $\langle A_i \mid i < \lambda\rangle$ *witnesses that* $\neg Gal(\mathcal{F}, \kappa, \lambda)$[2].

*Remark* 4.10. The idea of using Cohen subsets to generate a non-Glavin filter was already noticed in [3] but missed the part about extending the club filter.

*Proof.* We denote by $Add(\kappa, \lambda) := \{f : \lambda \times \kappa \to 2 \mid |f| < \kappa\}$ ordered as usual by inclusion. Note that the value $f(\langle i, \alpha\rangle) = 0, 1$ determined weather $\alpha \in A_i$ or $\alpha \notin A_i$ respectively. The proof the Cohen sets form a $\kappa$-independent family is well known and that the $\kappa$-complete filter generated by the Cohens is non-Galvin can be found in [3, Lemma 3.1]. Let us prove that given $I, J$ as above, $(\bigcap_{i\in I} A_i) \cap (\bigcap_{j\in J} A_j)$ is stationary in $V[\langle A_i \mid i < \lambda\rangle]$. Let $\underset{\sim}{C}$ be a $Add(\kappa, \lambda)$-name for a club in $\kappa$ and $f_0 \in Add(\kappa, \lambda)$ be any condition. Fix $N \prec H(\theta)$ an elementary submodel of some large enough $\theta$ such that:

1. $f_0, \underset{\sim}{C}, I, J, Add(\kappa, \lambda) \in N$.
2. $I, J \subseteq N$.
3. $|N| < \kappa$.
4. $\delta^* := N \cap \kappa$ is an ordinal.
5. For every $f \in N \cap Add(\kappa, \lambda)$, $\mathrm{dom}(f) \subseteq N$.

Such an elementary submodel is obtained using the usual Löwenheim–Skolem theorems to construct an increasing (with respect to inclusion) sequence of elementary submodels $N_n$, each $N_n$ has cardinality $< \kappa$, such that

$$f_0, \underset{\sim}{C}, I, J, Add(\kappa, \lambda) \in N_0$$

$\kappa \cap N_n \subseteq N_{n+1}$ and $\cup_{f\in N_n} \mathrm{dom}(f) \subseteq N_{n+1}$. Then the desired model is just $N_\omega = \cup_{m<\omega} N_n$. By the closure of $Add(\kappa, \lambda)$, we can find $f_0 \le f^* \in Add(\kappa, \lambda)$ which is $(N, Add(\kappa, \lambda))$-generic and $\mathrm{dom}(f^*) = \lambda \times \kappa \cap N$, namely, for every dense open $D \in N$, there is $f \in D \cap N$ such that $f \le f^*$. Let us prove that $f \Vdash \delta^* \in \underset{\sim}{C}$. Indeed for every $\rho < \delta^*$, $\rho \in N$. Hence we have

$$D_\rho := \{g \in Add(\kappa, \lambda) \mid \exists \delta \ge \rho. g \Vdash \delta \in \underset{\sim}{C}\}$$

definable in $N$ and it is clearly dense open. By genericity there is $f_\rho \le f^*$ such that $f_\rho \in N \cap D_\rho$. It follows that $N \models \exists \delta \ge \rho. f_\rho \Vdash \delta \in \underset{\sim}{C}$ and we can pick some $\rho \le \delta \in N \cap \kappa = \delta^*$ such that $f^* \Vdash \delta \in \underset{\sim}{C}$. So $f^* \Vdash \underset{\sim}{C} \cap \delta^*$ is unbounded in $\delta^*$ and since $\underset{\sim}{C}$ is a name for a club, $f^* \Vdash \delta^* \in \underset{\sim}{C}$. Finally, note that $\delta^* \notin N$ so for every $i \in I, j \in J$, the pairs, $\langle i, \delta^*\rangle, \langle j, \delta^*\rangle \notin \mathrm{dom}(f^*)$, so we can define $f_*$ such that $\mathrm{dom}(f_*) = \mathrm{dom}(f^*) \cup [(I \cup J) \times \{\delta^*\}]$, $f_* \restriction \mathrm{dom}(f^*) = f^*$, for every $i \in I$, $f_*(\langle i, \delta^*\rangle) = 1$ and for $j \in J$, $f_*(\langle j, \delta^*\rangle) = 0$. By density there is such a condition $f_* \in G$ which forces that $(\bigcap_{i\in I} \underset{\sim}{A}_i) \cap (\bigcap_{j\in J} \underset{\sim}{A}^c_j) \cap \underset{\sim}{C} \neq \emptyset$. $\square$

## 5. Open questions and aknoledments

In this section we collect the open problems which appeared in the previous chapters and add a few more:

**Question 5.1.** Does UA imply that every $\kappa$-complete ultrafilter over $\kappa$ satisfies that Galvin property?

[2]The notation $Gal(\mathcal{F}, \kappa, \lambda)$ means that for any collections $\mathcal{A}$ of $\lambda$-many sets in $\mathcal{F}$ there is a sub collection $\mathcal{B} \subseteq \mathcal{A}$ consisting of $\kappa$-many sets such that $\cap\mathcal{B} \in \mathcal{F}$.

**Question 5.2.** Is there a Galvin ultrafilter which is not Rudin-Keisler equivalent to an ultrafilter of the form appearing in corollary 2.10?

The next question seeks smaller large cardinals than a supercompact for which we can prove the existence of a non-Galvin filter.

**Question 5.3.** For which large cardinal notions "$\phi(x)$" the following holds: $\phi(\kappa)$ implies the existence of a non-Galvin $\kappa$-complete ultrafilter over $\kappa$ such that $Cub_\kappa \subseteq U$?

In this paper we proved that $\phi(x)$ = "$x$ is supercompact" satisfies the above. We conjecture that strongly compact cardinals are sufficient and that the filter extension property is the crucial ingredient for the construction of such ultrafilters.

**Question 5.4.** Is it consistent to have a model where there is a regular cardinal $\kappa$ with no non-Galvin filters?

Of course, by our results, this should be a model where $\diamond(\kappa)$ fails.

5.1. **Acknowledgments.** We would like to thank Moti Gitik for many helpful ideas. Also, we thank Gabriel Goldberg for pointing out the characterization of ultrafilters in the canonical inner models up to a superstrong cardinal.

## References

1. Uri Abraham and Saharon Shelah, *On the Intersection of Closed Unbounded Sets*, The Journal of Symbolic Logic **51** (1986), no. 1, 180–189.
2. James E. Baumgartner and Andras Hajnal, *Polarized partition relations*, J. Symbolic Logic **66** (2001), no. 2, 811–821. MR MR1833480 (2002k:03068)
3. Tom Benhamou, Shimon Garti, Moti Gitik, and Alejandro Poveda, *Non-galvin filters*, submitted (2022), arXiv:2211.00116.
4. Tom Benhamou, Shimon Garti, and Alejandro Poveda, *Negating the Galvin Property*, Submitted (2021), arXiv:2112.13373.
5. ______, *Galvin's property at large cardinals and an application to partition calculus*, Submitted (2022), arXiv:2207.07401.
6. Tom Benhamou, Shimon Garti, and Saharon Shelah, *Kurepa Trees and The Failure of the Galvin Property*, Proceedings of the American Mathematical Society (2021), arXiv:2111.11823.
7. Tom Benhamou and Moti Gitik, *Intermediate Models of Magidor-Radin Forcing-Part II*, Annals of Pure and Applied Logic **173** (2022), 103107.
8. ______, *On Cohen and Prikry Forcing Notions*, Submitted (2022), arXiv:2204.02860.
9. Shimon Garti, *Weak Diamond and Galvin's Property*, Period. Math. Hungar. **74** (2017), no. 1, 128–136. MR 3604115
10. Shimon Garti, *Tiltan*, Comptes Rendus Mathematique **356** (2018), no. 4, 351–359.
11. Shimon Garti, Yair Hayut, Haim Horowitz, and Magidor Menachem, *Forcing Axioms and The Galvin number*, Period. Math. Hungar. (2021), to appear.
12. Gabriel Goldberg, *The ultrapower axiom*, Berlin, Boston:De Gruyter, 2022.
13. Yair Hayut, *A note on the normal filters extension property*, Proc. Amer. Math. Soc. **148** (2020), 3129–3133.
14. Thomas Jech, *Stationary Sets*, Handbook of set theory, Springer, 2010, pp. 93–128.
15. Akihiro Kanamori, *Ultrafilters over a measurable cardinal*, Annals of Mathematical Logic **10** (1976), 315–356.
16. Farmer Schlutzenberg, *Measures in mice*, arXiv: 1301.4702 (2013), PhD Thesis.
17. John R. Steel, *An outline of inner model theory*, Handbook of set theory, Springer, 2010, pp. 1595–1684.

(Benhamou) Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, USA

*Email address*: tomb@uic.edu